ELECTRON IMPACT ON HYDROGENIC BOUND ELECTRON

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ABSTRACT. Cross-section for the excitation of a hydrogenic bound electron by electron impact has been calculated in a Field Theoretic way. For low impact energy this reduces to the usual Born approximation result. At higher energies the result is a relativistic generalization. Better approximations for Bethes approximate formulae for high energy excitation cross-section are also obtained.

In the following discussion we consider the excitation of hydrogenic atom by electron impact in a Field Theoretic way (Roy, 1960). Bound state wave function of a hydrogen atom may be written in the form

$$\phi(x, x') = \langle V \mid \psi(x)\psi(x') \mid V \rangle \quad \ldots \quad (1)$$

where ϕ is an igen-state solution of the equation

$$-\frac{1}{i}\,\frac{\partial \phi}{\partial t} = \left(H_D + \frac{e^2}{|\vec{x}_1 - \vec{x}_2|}\right)\phi \quad \ldots \quad (2)$$

H_D is the Dirac Hamiltonian in the configuration space of the electron and the proton. With sufficient accuracy proton mass is taken to be infinitely heavy. Now we may write the initial and final wave functions for the bound electron in the form

$$\psi_i(x) = \int g_i(k_1) u_{k_1}(x) d\vec{k}_1$$

$$\psi_f(x) = \int g_f(\lambda_1) u_{\lambda_1}(x) d\lambda_1$$

Thus the wave function for a bound and a free electron is

$$u_p(x_2)\psi(x_1) = \int g(k_1)\delta(p - k_2) u_{k_1}(x_1) u_{k_2}(x_2) d\vec{k}_1 d\vec{k}_2 \quad (4)$$

Hence the initial and final state vectors are

$$|\psi_i\rangle = \int g_i(k_1)\delta(p_1 - k_2) a_{k_1}{}^* a_{k_2}{}^* d\vec{k}_1 d\vec{k}_2 \mid V\rangle \quad (5)$$

$$|\psi_f\rangle = \int g_f(\lambda_1)\delta(p'_1 - \lambda_2) a_{\lambda_1}{}^* a_{\lambda_2}{}^* d\vec{\lambda}_1 d\vec{\lambda}_2 \mid V\rangle$$

The matrix element for the excitation of the bound electron from the state ψ_i to ψ_f by an electron impact which in the process transisits from state '1' to '1''' is given by

$$<\psi_f | S | \psi_i> = -ie^2 \int [{}^{-}\psi_f(x_2)\gamma^\mu \psi_i(x_2) D_c(x_2 - x_1) \bar{u}p'_1(x_1)\gamma_\mu u p_1(x_1) d^4x_1 d^4x_2$$
$$-(1' \rightleftharpoons 2')] \quad \ldots \quad (6)$$

In momentum space the matrix element is

$$M_{fi} = (2\pi)^4 e^2 \int d\vec{p}[\bar{u}(p'_1)\gamma^\mu u(p_1)\bar{V}_f(p)\gamma_\mu V_i(q) - \bar{V}_f(p)\gamma_\nu u(p_1)\bar{u}(p'_1)\gamma^\nu V_i(q)] \quad \ldots \quad (7)$$

where

$$P = (W_2', p)$$

$$q = p + p_1' - p_1$$

$V_i(p)$ and $V_f(p)$ are the Fourier transforms of bound state wave functions for the initial and final states.

i.e.

$$\psi_i(r) = \int V_i(p) e^{ip \cdot r} dp$$

$$\psi_f(\vec{r}) = \int V_f(p) e^{i\vec{p} \cdot \vec{r}} d\vec{p} \quad \ldots \quad (8)$$

Non-relativistic approximation for the bound wave function :

In this approximation we used non-relativistic Schrodinger wave function. So we take

$$V(p) = \phi_{nlm}(p)\, U(0)$$

where $\vec{U(s)}$ stands for the plane wave spinor for momentum $\vec{s}$. Here we take ϕ_{nlm} and $\phi_{n'l'm'}$ corresponding to the initial and final states. We sum over final spins and average over initial spins for the square of the matrix element and get

$$|\bar{M}_{fi}|^2 = \int_f \bar{\int_i} |M_{fi}|^2 = \frac{(2\pi)^8 e^4}{4} \int d\vec{p} d\vec{\bar{p}} \Big[\frac{A}{(P_1 - P'_1)^4} + \frac{B}{(P_1 - P)^2 (P_1 - \bar{P})^2}$$

$$- \frac{C}{(P_1 - P'_1)^2 (P_1 - P)^2} - \frac{D}{(P_1 - P'_1)^2 (P_1 - \bar{P})^2} \cdot \Big] \quad (9)$$

where

$$A = F.\mathrm{Tr}\{\gamma^\mu \Lambda(p_1')\gamma^\nu \Lambda(p_1)\}\, \mathrm{Tr}\,\{\gamma_\mu \Lambda(0)\, \gamma_\nu \Lambda((0)\}$$

$$B = F.\mathrm{Tr}\,\{\gamma^\mu \Lambda(p'_1)\gamma^\nu \Lambda(0)\}\, \mathrm{Tr}\,\{\gamma_\mu \Lambda(0)\gamma_\nu \Lambda(p_1)\}$$

$$C = F.\mathrm{Tr}\,\{\gamma^\mu \Lambda(0)\, \gamma_\nu \Lambda(p_1)\gamma_\mu \Lambda(p'_1)\gamma^0 \Lambda(0)\}$$

$$D = F.\mathrm{Tr}\{\gamma^\mu \Lambda(p_1')\gamma^\nu \Lambda(p_1)\, \gamma_\nu \Lambda(0)\gamma_\mu \Lambda(0)\}$$

where

$$F = \phi_i(q)\phi_i^*(\bar{q})\phi_f(\bar{p})\phi_f^*(p) \quad (10)$$

On evaluation of the traces

$$A = F.\ \frac{1}{m^2}\ [4W_1W'_1+2(m^2-p_1p'_1)]$$

$$B = F.\ \frac{1}{m^2}\ [2W_1W'_1+2p_1{\cdot}p'_1-2mW_1-2mW'_1+4m^2] \qquad \dots \quad (11)$$

$$C = D = F.\ \frac{1}{m^2}\ [(p_1\,p'_1)+2mW_1+2mW'_1-2W_1W'_1-m^2]$$

Excitation cross-section is given by

$$d\sigma = \frac{(2\pi)^2}{(2\pi)^{12}} d\Omega'_1\ W_1 p'_1\ W'_1\,dW'_1\,|\,M_{fi}\,|^2\delta(W_1+W_2-W'_1-W'_2) \dots \quad (12)$$

Integration with respect to W'_1 yields

$$\frac{d\sigma}{d\Omega'_1} = \left(\frac{e^2}{4\pi}\right)^2 \frac{W_1\,W_1'}{p_1}\ p'_1 \left\{\frac{A'I_1^2}{(P_1-P'_1)^4} + B'I_2^2 - \frac{+2C'I_1I_2}{(P_1-P'_1)^2}\right\} \quad \dots \quad (13)$$

where $\qquad A' = \frac{1}{m^2}\{4W_1W'_1+2(m^2-p_1{\cdot}p_1')\}$ $\qquad$ etc.

$$I_1 = \int \phi_i\ (\vec{q})\phi_f\ (\vec{p})d\vec{p} \qquad \dots \quad (14)$$

and

$$I_2 = \int \frac{\phi_i(\vec{q})\phi_f(\vec{p})}{(P_1-P)^2} d\vec{p} \qquad \dots \quad (15)$$

The following cases are now considered :

(1) *Low energy scattering*

For very low impact energy (near the threshold energy) we can confidently neglect higher powers of momenta. In this approximation A', $B' \to 4$ and $C' = D' \to 2$. The result is nothing but the non-relativistic Born term. The merits and demerits of these results have been investigated by many authors. Corinaldesi and Trainer (1952) have evaluated the integrals analytically for special values of nl and $n'l'$.

(2) *Moderately high impact energy*

In this case the nature of the cross-section may well be investigated from the formulae (13), (14) and (15) with the further replacement of $(P_1-P)^2$ by $-(\vec{p}_1-\vec{p})^2$ in expression (15). In this approximation I_1 and I_2 corresponds to integrals for irect and exchange amplitudes. Tables (Omidvar, 1965) for integrals I_1 and I_2 for different inidal and final quantum numbers are available and hence it is a simple matter totifind corresponding cross-sections using formula (13).

(3) *High energy cross-section for direct scattering*

For large impact energies the exchange term becomes negligible and the nature of the scattering cross-section is determined solely by the direct term.

Differential cross-section is

$$\frac{d\sigma}{d\Omega'_1} = \left(\frac{e^2}{4\pi}\right)^2 \frac{W_1 W'_1 p'_1}{p_1 m^2} \{4W_1 W'_1 + 2(m^2 - p_1.p'_1)\} \frac{I_1^2}{(P_1 - P'_1)^4} \quad \ldots (16)$$

For high impact energy $W'_1 \sim W_1$ and $t = |\vec{p}_1 - \vec{p'}_1|$ is very small, of the order W^2_{ab}/p_1 for $\theta \to 0$. Most of the contribution come but from a very small angle about the forward direction. Bethe's (1950) approximation formulae are then slightly modified. These take the forms

$$\sigma_{Tot} \sim \pi a_0^2 \left\{ \frac{\pi W_1^2 (6W_1^2 - 2m^2)}{p_1^2 m^2} \right\} \left[\log \left(\frac{K^2_T - W^2_{ab}}{K^2_{min} - W^2_{ab}{}^1} \right) + \frac{W^2_{ab}}{(K_{min}^2 - W^2_{ab})} \right] |M^1_{12}|^2 \quad \ldots (17)$$

for optically allowed transition, and

$$\sigma_{Tof} \sim \pi a_0^2 \frac{\pi W_1^2}{4p_1^2} (6W^2_1 - 2m^2) \left[(K^2_T - K^2_{min}) + 2W^2_{ab} \log \left(\frac{K^2_T - W^2_{ab}}{K^2_{min} - W^2_{ab}} \right) + \frac{W^4_{ab}}{(K^2_{min} - W^2_{ab}} \right] |M^2_{12}|^2 \quad \ldots (18)$$

for optically forbidden transition.

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